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EVERY ODD PERFECT NUMBER HAS A PRIME FACTOR WHICH EXCEEDS 10^6

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ABSTRACT. It is proved here that every odd perfect number is divisible by a prime greater than 10^6.

1. INTRODUCTION

In what follows, a, b, c, ... will be used to represent non-negative integers, with primes being symbolized by p, q and r. An element of the (possibly empty) set of odd perfect numbers will be denoted by N, so that $\sigma(N) = 2N$ where σ is the familiar sum of divisors function. The dth cyclotomic polynomial will be symbolized by F_d, so that $F_p(x) = 1 + x + x^2 + \cdots + x^{p-1}$. If p and m are relatively prime, $h(p;m)$ will denote the order of p modulo m.

According to Theorem 3.4 in [9],

(1) $\sigma(p^a) = \prod_d F_d(p)$, where $d \mid (a+1)$ and $d > 1$.

From Theorems 94 and 95 in [8],

(2) $q \mid F_m(p)$ if and only if $m = q^b h(p;q)$. If $b > 0$, then $q \parallel F_m(p)$. If $b = 0$, then $q \equiv 1 \pmod{m}$.

It follows from (2) that

(3) if $q \mid F_r(p)$, then either $r = q$ and $p \equiv 1 \pmod{q}$ (and $q \parallel F_r(p)$) or $q \equiv 1 \pmod{r}$;

(4) if $q = 3$ or 5 and $m > 1$ is odd, then $q \mid F_m(p)$ (in fact, $q \parallel F_m(p)$) if and only if $m = q^b$ and $p \equiv 1 \pmod{q}$.

According to a result due to Bang [1],

(5) if p is an odd prime and $m \geq 3$, then $F_m(p)$ has at least one prime factor q such that $q \equiv 1 \pmod{m}$.

It is well known, and easy to prove, that

(6) $N = p_0^{a_0} p_1^{a_1} \dots p_u^{a_u}$, where the p_i are distinct odd primes, $p_0 \equiv a_0 \equiv 1 \pmod{4}$, and $2 \mid a_i$ if $i > 0$. (In this, p_0 is called the *special* prime.)

In [4], it was proved that at least one of the p_i in (6) exceeds 100110. In 1978, Condict [3], in his senior thesis at Middlebury College, improved this bound to 300000, while in 1982 Brandstein [2] announced that at least one of the p_i is greater than 500000. (To the best of our knowledge, however, Brandstein's announcement has never been substantiated by a public proof.) The purpose of the present paper is to improve these results by proving the following

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Theorem. *If N is odd and perfect, then N has a prime factor which exceeds* 10^6.

Our proof will be by contradiction. Thus, we now assume without further explicit mention that $p_i < 10^6$ for every p_i in (6), and shall show that this assumption is untenable.

Since N is perfect, and since σ is multiplicative, it follows from (1) and (6) that

(7) $2N = \prod_{i=0}^{u} \sigma(p_i^{a_i}) = \prod_{i=0}^{u} \prod_d F_d(p_i)$, where $d \mid (a_i+1)$ and $d > 1$ (with $d = 2$ if and only if $i = 0$).

We see immediately that the set of p_i in (6) is identical with the set of odd prime factors of the $F_d(p_i)$ in (7). In particular, recalling our assumption we note that all of the prime factors of each $F_d(p_i)$ must be less than 10^6. Our proof will hinge on the consequence that if r is a prime divisor of $a_i + 1$, then every prime factor of $F_r(p_i)$ must be less than 10^6.

2. Acceptable values of $F_r(p)$

If $p > 2$ and r are primes, we shall say that $F_r(p)$ is *acceptable* if every prime divisor of $F_r(p)$ is less than 10^6. It follows easily from (5) that if $r > 500000$, then $F_r(p)$ is unacceptable for every odd prime p. We shall say that the prime p is *inadmissible* if $F_r(p)$ is unacceptable for every prime r (with $r = 2$ taken into consideration only if it is possible that p is the special prime for N).

An extensive computer search revealed that if $3 \le p < 10^6$ and $r \ge 7$, then $F_r(p)$ is unacceptable except for the 35 pairs of values of p and r listed in Table 1. Details of the search and supporting arguments may be found in an appendix to this paper. At the suggestion of a referee, some of these arguments have been included in Section 7 of the present paper. The complete appendix appears in [5] and is available upon request from the second author.

3. An important set of primes

Our objective in this section is to show that N is not divisible by certain "small" primes.

Lemma 1. *Let X be the set of primes*

$$X = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 43, 61, 127, 131, 151, 1093\}.$$

If $p \in X$, then $p \nmid N$.

The proof proceeds by considering each prime p in X in turn, but in the order

$$1093, 151, 31, 127, 19, 11, 7, 23, 131, 37, 61, 13, 3, 5, 29, 43, 17.$$

We assume $p \mid N$ and find all acceptable values of $F_r(p)$ (with $r = 2$ being considered only if p might be the special prime); from (7), $F_r(p) \mid 2N$ for at least one acceptable $F_r(p)$ and each odd prime divisor of this $F_r(p)$ divides N; from each acceptable $F_r(p)$ a single odd prime divisor, say q, is selected and all of the acceptable values of $F_r(q)$ are determined. This procedure is iterated until an inadmissible prime or some other contradiction is encountered, thus showing that $p \nmid N$. Treating the primes of X in the given order allows those already considered to be used in the elimination of subsequent ones.

We shall illustrate the method by showing that $1093 \nmid N$ and $151 \nmid N$. The complete proof of Lemma 1 is given in the appendix mentioned in Section 2. Hopefully, the nomenclature we use is self-explanatory. We write p^* to imply that p is the

TABLE 1. Acceptable values of $F_r(p)$ for $3 \le p < 10^6$ and $r \ge 7$

p	r	$F_r(p)$
3	7	1093
3	11	$23 \cdot 3851$
3	13	797161
3	17	$1871 \cdot 34511$
3	19	$1597 \cdot 363889$
5	7	19531
7	7	$29 \cdot 4733$
7	11	$1123 \cdot 293459$
11	7	$43 \cdot 45319$
13	11	$23 \cdot 419 \cdot 859 \cdot 18041$
19	7	$701 \cdot 70841$
43	7	$7 \cdot 5839 \cdot 158341$
59	7	$43 \cdot 281 \cdot 757 \cdot 4691$
67	7	$175897 \cdot 522061$
79	7	$281 \cdot 337 \cdot 1289 \cdot 2017$
127	7	$7 \cdot 43 \cdot 86353 \cdot 162709$
131	7	$127 \cdot 189967 \cdot 211093$
191	7	$127 \cdot 197 \cdot 10627 \cdot 183569$
269	7	$43 \cdot 211 \cdot 631 \cdot 2633 \cdot 25229$
359	7	$211 \cdot 449 \cdot 1303 \cdot 4019 \cdot 4327$
389	7	$127 \cdot 337 \cdot 659 \cdot 827 \cdot 148933$
431	7	$29 \cdot 953 \cdot 967 \cdot 1009 \cdot 238267$
2381	7	$7 \cdot 43 \cdot 2689 \cdot 3613 \cdot 72997 \cdot 853903$
2713	7	$29^2 \cdot 43 \cdot 73361 \cdot 258469 \cdot 581729$
3301	7	$29^2 \cdot 911 \cdot 38669 \cdot 186733 \cdot 233941$
3779	7	$197 \cdot 2311 \cdot 23773 \cdot 455407 \cdot 591053$
4327	7	$7 \cdot 3221 \cdot 5503 \cdot 5657 \cdot 92401 \cdot 101221$
8009	7	$7 \cdot 43 \cdot 127 \cdot 491 \cdot 127247 \cdot 305873 \cdot 361313$
9719	7	$281 \cdot 3067 \cdot 8219 \cdot 19937 \cdot 30773 \cdot 193957$
10889	7	$2003 \cdot 22093 \cdot 116341 \cdot 471997 \cdot 686057$
10949	7	$7 \cdot 29 \cdot 197 \cdot 547 \cdot 1009 \cdot 6917 \cdot 25523 \cdot 442177$
27457	7	$29 \cdot 42463 \cdot 65171 \cdot 71261 \cdot 91813 \cdot 816047$
53831	7	$7 \cdot 29 \cdot 39341 \cdot 104651 \cdot 257489 \cdot 269221 \cdot 420001$
191693	7	$7561 \cdot 11887 \cdot 14869 \cdot 16759 \cdot 89839 \cdot 118399 \cdot 208279$
493397	7	$29^2 \cdot 127 \cdot 1163 \cdot 2129 \cdot 4229 \cdot 26041 \cdot 50177 \cdot 71359 \cdot 138349$

special prime. (Of course, two different primes cannot both be special simultaneously.)

A. $1093 \nmid N$.

A, 1093: $F_2(1093) = 2 \cdot 547$; $F_3(1093) = 3 \cdot 398581$.

A, 1093*, 547: $F_3(547) = 3 \cdot 163 \cdot 613$.

A, 1093*, 547, 613: $F_3(613) = 3 \cdot 7 \cdot 17923$; $F_5(613) = 131 \cdot 20161 \cdot 53551$.

A, 1093*, 547, 613, 17923: $F_3(17923) = 3 \cdot 13 \cdot 31 \cdot 265717$.

A, 1093*, 547, 613, 17923, 265717: 265717 is inadmissible.

A, 1093*, 547, 613, 20161: 20161 is inadmissible.

A, 1093, 398581: $F_2(398581) = 2 \cdot 17 \cdot 19 \cdot 617$.

A, 1093, 398581*, 617: $F_3(617) = 97 \cdot 3931$.

A, 1093, 398581*, 617, 3931: $F_3(3931) = 3 \cdot 7 \cdot 31 \cdot 23743$.

A, 1093, 398581*, 617, 3931, 23743: 23743 is inadmissible.

B. $151 \nmid N$.

B, 151: $F_3(151) = 3 \cdot 7 \cdot 1093$, contradiction to A.

4. A restriction on the exponents in the prime power decomposition of N

Suppose that $p^a \parallel N$ and $r \mid (a+1)$ where $r > 5$. Then $F_r(p)$ appears in Table 1 and, from (7), $F_r(p) \mid N$. It follows from Table 1 and Lemma 1 that $r = 7$ and $p \in \{67, 79, 359, 3779, 9719, 10889, 191693\}$. Referring to Table 1, we see that if $p = 67$, then $175897 \mid N$. Since $F_r(175897)$ is acceptable only for $r = 2$ and since $F_2(175897) = 2 \cdot 37 \cdot 2377$ (and $37 \nmid N$ from Lemma 1), we conclude that $p \neq 67$. Similarly, if $p = 79$, then $337 \mid N$; only $F_2(337) = 2 \cdot 13^2$ and $F_3(337) = 3 \cdot 43 \cdot 883$ are acceptable and since neither 13 nor 3 divides N we see that $p \neq 79$. If $p = 359$, then $1303 \mid N$; only $F_3(1303) = 3 \cdot 13 \cdot 19 \cdot 2293$ is acceptable and $3 \nmid N$, so $p \neq 359$. If $p = 3779$, then $455407 \mid N$; since 455407 is inadmissible, $p \neq 3779$. If $p = 9719$, then $3067 \mid N$; since only $F_3(3067) = 3 \cdot 127 \cdot 24697$ is acceptable and $3 \nmid N$, we see that $p \neq 9719$. If $p = 10889$, then $471997 \mid N$; but only $F_2(471997) = 2 \cdot 19 \cdot 12421$ is acceptable and $19 \nmid N$, so $p \neq 10889$. Finally, $p \neq 191693$ since otherwise $11887 \mid N$, and 11887 is inadmissible.

We have proved

Lemma 2. *If $p^a \parallel N$ and p is not the special prime p_0, then $a + 1 = 3^b \cdot 5^c$ where $b + c > 0$. If $p_0^{a_0} \parallel N$, then $a_0 + 1 = 2 \cdot 3^b \cdot 5^c$ where $b + c \geq 0$.*

5. Four important sets

Let $S = \{47, 53, 59, \ldots\}$ be the set of all primes p such that $p \not\equiv 1 \pmod 3$, $p \not\equiv 1 \pmod 5$ and $37 < p < 10^6$. It follows from Lemma 2, (7) and (2) that if $p \in S$ and $p \nmid F_2(p_0)$, then $p \nmid N$. (For if $p \mid F_d(p_i)$ and $d \neq 2$ in (7), then either $3 \mid d$ and then $p \equiv 1 \pmod 3$, or $5 \mid d$ and then $p \equiv 1 \pmod 5$; so $p \notin S$.) At most one element of S can divide $F_2(p_0)$. For suppose that $p_i \in S$ and $p_i^{a_i} \parallel N$ and $p_i \mid F_2(p_0)$. Then $p_i^{a_i} \parallel F_2(p_0)$, and if two elements of S were divisors of $F_2(p_0)$ it would follow that $F_2(p_0) = p_0 + 1 \geq 2 \cdot 47^2 \cdot 53^2 > 12 \cdot 10^6$. This is impossible since $p_0 < 10^6$. Note that $p_0 \notin S$ since otherwise $3 \mid F_2(p_0)$, so $3 \mid N$ in contradiction to Lemma 1.

We have proved

Proposition 1. *The number N is divisible by at most one element of S. (If there is such an element s, then $s \neq p_0$ and $s \geq 47$.)*

Computer searches showed that S has 29451 elements, and

(8) $S^* = \prod_{p \in S} p/(p-1) > 1.6358.$

Let $T = \{61, 151, 181, \ldots\}$ be the set of all primes p such that $p \equiv 1 \pmod{15}$ and $37 < p < 10^6$. It follows from Lemma 2, (7) and (4) that if $p \in T$ and $p \neq p_0$, then $p \nmid N$. (For if $p_i \in T$ and $p_i^{a_i} \parallel N$ where $i > 0$, then $3 \mid (a_i + 1)$ or $5 \mid (a_i + 1)$, so that $F_3(p_i) \mid N$ and then $3 \mid N$, or $F_5(p_i) \mid N$ and then $5 \mid N$, either of which contradicts Lemma 1.)

We have proved

Proposition 2. *The number N is divisible by at most one element of T. (If there is such an element it is p_0, and then $p_0 \geq 61$.)*

Computer searches showed that T has 9806 elements, and

(9) $T^* = \prod_{p \in T} p/(p-1) > 1.1567.$

Now, let $U = \{43, 73, 79, \dots\}$ be the set of all primes p such that $p \equiv 1 \pmod 3$, $p \not\equiv 1 \pmod 5$, $F_5(p)$ has a prime factor which exceeds 10^6, and $37 < p < 10^6$. It follows from Lemma 2, (7) and (4) that if $p \in U$ and $p \neq p_0$, then $p \nmid N$. (For if $p_i \in U$ and $p_i^{a_i} \parallel N$ where $i > 0$, then $3 \mid (a_i + 1)$ and $F_3(p_i) \mid N$ and $3 \mid N$, or $5 \mid (a_i + 1)$ and $F_5(p_i) \mid N$ and N has a prime factor which exceeds 10^6. In either case we have a contradiction.)

We have proved

Proposition 3. *The number N is divisible by at most one element of U. (If there is such an element it is p_0, and then $p_0 \geq 73$.)*

Computer searches showed that U has 29115 elements, and

(10) $U^* = \prod_{p \in U} p/(p-1) > 1.4919.$

Finally, let $V = \{1091, 1181, 1811, \dots\}$ be the set of all primes p such that $p \equiv 1 \pmod 5$, $p \not\equiv 1 \pmod 3$, $F_3(p)$ has a prime factor which exceeds 10^6, and $37 < p < 10^6$. If $p \in V$, then $p \neq p_0$, since $3 \mid F_2(p)$, and it follows from Lemma 2, (7) and (4) that $p \nmid N$.

We have proved

Proposition 4. *The number N is not divisible by any element of V.*

Computer searches showed that V has 6719 elements, and

(11) $V^* = \prod_{p \in V} p/(p-1) > 1.0389.$

Note that S, T, U and V are pairwise disjoint.

6. The proof of our theorem

There are 78486 primes p such that $37 < p < 10^6$, and

(12) $P^* = \prod_{41 \leq p < 10^6} p/(p-1) < 3.6597.$

If $p^a \parallel N$, then $1 < \sigma(p^a)/p^a < p/(p-1)$. Since σ is a multiplicative function and $x/(x-1)$ is monotonic decreasing for $x > 1$, it follows from Lemma 1 (using here only that $p \nmid N$ for $p \leq 37$), Propositions 1, 2, 3, 4 and (7), (8), (9), (10), (11), (12) that

$$2 = \frac{\sigma(N)}{N} < \prod_{i=0}^{u} \frac{p_i}{p_i - 1} \leq \frac{47}{46}\,\frac{61}{60}\,\frac{P^*}{S^*T^*U^*V^*} < 1.2963.$$

(Note that 47 and 61 appear explicitly due to Propositions 1 and 2.) This contradiction proves our theorem.

7. Some details on the search for acceptable values of $F_r(p)$

As can be seen from Table 1, if $3 \leq p < 10^6$ and $r \geq 7$, only 35 values of $F_r(p)$ are acceptable. (Of course, it follows from (5) that $F_r(p)$ is unacceptable if $r > 500000$.) In establishing this fact it was essential that those $F_r(p)$ be determined which are divisible by at least the second power of a prime. The tables to be found in [6] and [7] were helpful in this regard, but their ranges were much too narrow for most of our searches.

Suppose first that $701 \le r < 500000$ and $10^2 < p < 10^6$. A computer search showed that

(i) if $q < 10^6$, then $q^3 \nmid F_r(p)$ except that $3119^3 \parallel F_{1559}(146917)$;
(ii) there are at most 116 primes q such that $q < 10^6$ and $q \equiv 1 \pmod r$ (and, specifically, there are 116 primes less than 10^6 and congruent to 1 modulo 751), except that there are exactly 122 primes less than 10^6 and congruent to 1 modulo 719 (13 of which, including 1439, are less than 10^5 and 109 of which are between 10^5 and 10^6).

Now suppose that $r \ge 701$, $10^2 < p < 10^6$ and all of the prime factors of $F_r(p)$ are less than 10^6. Then, from (5), $r < 500000$. If $r = 719$, then $F_r(p) > p^{r-1} > (10^2)^{718} = 10^{1436}$; but, from (3), (i) and (ii), $F_{719}(p) < 1439^2((10^5)^2)^{12}((10^6)^2)^{109} < 10^{1435}$. If $r \ne 719$, then $F_r(p) > p^{r-1} > (10^2)^{700} = 10^{1400}$; but, from (3), (i) and (ii), $F_r(p) < ((10^6)^2)^{116} = 10^{1392}$ (where, in particular, $F_{1559}(146917) < 3119^3((10^6)^2)^{56} < 10^{683}$, since there are exactly 57 primes less than 10^6 which are congruent to 1 modulo 1559). These contradictions yield

Proposition 5. *If $r \ge 701$ and $10^2 < p < 10^6$, then $F_r(p)$ has a prime factor which exceeds 10^6.*

Next, suppose that $487 \le r < 701$ and $10^2 < p < 10^6$. A computer search showed that

(iii) if $q < 10^6$, then $q^3 \nmid F_r(p)$ and $q^2 \parallel F_r(p)$ for at most one such q (and a fixed value of p);
(iv) there are at most 163 primes q such that $q < 10^6$ and $q \equiv 1 \pmod r$ (and, specifically, there are 163 primes less than 10^6 and congruent to 1 modulo 499).

Now suppose in addition that all of the prime factors of $F_r(p)$ are less than 10^6. If $r \ge 499$, then $F_r(p) > p^{r-1} > (10^2)^{498} = 10^{996}$; but, from (3), (iii) and (iv), $F_r(p) < (10^6)^2(10^6)^{162} = 10^{984}$. If $r = 487$ or 491 then, since there are exactly 156 primes less than 10^6 and congruent to 1 modulo 487 and exactly 153 primes less than 10^6 and congruent to 1 modulo 491, we see that $F_r(p) > (10^2)^{486} = 10^{972}$ and $F_r(p) < (10^6)^2(10^6)^{155} = 10^{942}$. These contradictions prove

Proposition 6. *If $487 \le r < 701$ and $10^2 < p < 10^6$, then $F_r(p)$ has a prime factor which exceeds 10^6.*

A slightly more complicated argument yields

Proposition 7. *If $7 \le r < 487$ and $10^2 < p < 10^6$, then $F_r(p)$ has a prime factor which exceeds 10^6, except for $r = 7$ and the values of p (exceeding 10^2) listed in Table 1.*

It remains to consider those $F_r(p)$ for which $r \ge 7$ and $p < 10^2$. According to the table in [6], $48947^2 \parallel F_{24473}(17)$, $47^2 \parallel F_{23}(53)$, $59^2 \parallel F_{29}(53)$, $47^2 \parallel F_{23}(71)$ and $4871^2 \parallel F_{487}(83)$; otherwise, if $q^2 \mid F_r(p)$ where $r > 5$ and $p < 10^2$, then $q > 10^6$. In each of the five exceptional cases just mentioned, $F_r(p)$ has a prime factor which exceeds 10^6 and is therefore unacceptable. The study of the remaining cases, in each of which either $F_r(p)$ is divisible by a prime greater than 10^6 or $F_r(p)$ is squarefree, yields the first 15 entries in Table 1 and no other acceptable values of $F_r(p)$. The details of this study are omitted here.

8. Concluding remarks

Let Q be the largest prime factor of the odd perfect number N. We have shown that $Q > 10^6$. A referee has pointed out that our proof could probably be modified so as to improve the lower bound on Q from 10^6 to 10^7. He/she is undoubtedly correct. However, the time and effort to do so seem prohibitive to the present authors at the present time for the following reasons.

Our target of 10^6 in this paper was largely determined by the fact that a list of all 78498 primes up to 10^6 was already available for use in memory in the CYBER 860 at the Temple University Computing Center. Using the procedures of this paper, to show that $Q > 10^7$ would necessitate having in memory a list of all 664579 primes up to 10^7.

Now, let $\pi(x)$ denote the number of primes which do not exceed the real number x. We have $\pi(10^6) = 78498$ and $\pi(5 \cdot 10^5) = 41538$. It follows that, in the construction of Table 1, $78497 \cdot 41535 = 3260372895 = P_1$ values of $F_r(p)$ had to be examined for acceptability (taking $r < 500000$ since otherwise $F_r(p)$ is unacceptable). The searches involved in generating Table 1 of this paper required approximately 450 hours of time on the CYBER 860 and perhaps an additional 250 hours (we did not keep accurate records) of time on a 486 PC. Suppose now that the definition of "acceptability" were changed to: "$F_r(p)$ is acceptable if every prime divisor of $F_r(p)$ is less than 10^7." Since $\pi(10^7) = 664579$ and $\pi(5 \cdot 10^6) = 348513$, if Table 1 were now to be regenerated for $3 \le p < 10^7$ and $r \ge 7$, then $664578 \cdot 348510 = 231612078780 = P_2$ values of $F_r(p)$ would have to be investigated for acceptability. Each such investigation would require at least as much time as those undertaken in the present paper. Therefore, since $P_2/P_1 > 71$, it seems rather conservative to anticipate spending around 30000 hours of time on the CYBER 860 in the generation of Table 1 if one wished to prove that $Q > 10^7$. This estimate is sufficient to discourage the present authors from making such an attempt.

The same referee has also remarked that the contradictory inequality established in Section 6 is much stronger than is needed and that our theorem could be proved using only the sets S and U. This is true, but we have chosen not to omit the sets T and V from consideration since, as the referee says, "the overkill in the inequality in Section 6 partially substantiates" his/her (and our) feeling that a higher lower bound on Q is achievable by the methods of this paper.

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